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# NONNEGATIVE AND SKEW-SYMMETRIC PERTURBATIONS OF A MATRIX WITH POSITIVE INVERSE

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ABSTRACT. Let $A$ be a nonsingular matrix with positive inverse and $B$ a nonnegative matrix. Let the inverse of $A+vB$ be positive for $0 \le v < v^* < +\infty$ and at least one of its entries be equal to zero for $v = v^*$; an algorithm to compute $v^*$ is described in this paper. Furthermore, it is shown that if $A + A^{\mathrm{T}}$ is positive definite, then the inverse of $A+v(B-B^{\mathrm{T}})$ is positive for $0 \le v < v^*$.

## 1. INTRODUCTION

Let

$$A + vB \tag{1}$$

be an $n \times n$ real matrix, where $A$ is a nonsingular matrix with positive inverse ([5, 2, 1]), $B$ ($B \neq 0$) a nonnegative matrix and $v$ a nonnegative real parameter,

$$A^{-1} > 0, \quad B \ge 0, \quad B \neq 0, \quad v \ge 0. \tag{2}$$

The parameter $v$ may be considered as a measure of the size of the nonnegative perturbation $vB$ of the matrix $A$. Let

$$Z(v) = (A + vB)^{-1} = [z_{ij}(v)]. \tag{3}$$

For $v = 0$, we have $Z(0) = A^{-1} > 0$; thus, $\det(A + vB) \neq 0$ and $Z(v) > 0$ in a sufficiently small neighborhood of 0. This paper addresses the problem of finding the largest, possibly infinite, number $v^*$ such that $A + vB$ is nonsingular and $Z(v) > 0$ in $[0, v^*)$. We will describe an algorithm (the iterative process (6)) to compute $v^*$ if $v^* < +\infty$. In the case $v^* = +\infty$, the successive approximations defined by (6) form a sequence diverging monotonically to $+\infty$.

We shall consider also matrices of the type

$$C(v) = A + v(B - B^{\mathrm{T}}); \tag{4}$$

here the matrix $A$ is perturbed by a skew-symmetric matrix which may be written as $B - B^{\mathrm{T}}$ with $B \ge 0$. It will be shown that if $A + A^{\mathrm{T}}$ is positive definite, then $C^{-1}(v) \ge Z(v) > 0$ in $[0, v^*)$, where $Z$ is defined by (3).

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Numerical calculations have been performed by using the matrix involved in the discrete analog of the integro-differential equation

$$(5) \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[p\frac{\partial u}{\partial x}\right] + q[u_0 - u] + v\int_0^1 K(x, x')[u_0(x') - u(x')]\,dx'$$

with boundary conditions $u(0) = u(1) = 0$, where $p(x) > 0$, $q(x) \geq 0$, $u_0(x) \geq 0$, and $K(x, x') \geq 0$. Equation (5) is a model for a spatially distributed community whose migration has both a random and a special deterministic component; more complicated models ( $n$-species communities, nonlinear) can be obtained including birth-death processes, competition and predator-prey interactions [4]. A direct finite difference approach to (5) provides a discrete approximation $\mathbf{u}$ of the steady state solution $u$ satisfying an equation of the type $(A + vB)\mathbf{u} = \mathbf{f} \geq 0$, where $A + vB$ is of type (2); the positivity of its inverse assures the positivity and the stability of $\mathbf{u}$.

## 2. The inverse of $A + vB$

**Lemma 1.** *Assume* (2) *and let* $\det(A + vB) \neq 0$ *and* $Z(v) > 0$, *with* $Z(v)$ *as defined in* (3). *Then* $Z'(v) < 0$ *and* $Z''(v) > 0$.

*Proof.* From the identity $(A + vB)Z(v) = I$ we obtain

$$Z' = -ZBZ, \qquad Z'' = -2ZBZ' = 2ZBZBZ,$$

where $Z' = dZ/dv = [z'_{ij}]$ and $Z'' = dZ'/dv = [z''_{ij}]$. As $B \geq 0$, $B \neq 0$, and $Z(v) > 0$, there follows $Z'(v) < 0$ and $Z''(v) > 0$. □

**Lemma 2.** *Under the assumptions of Lemma* 1, *let* $v_\alpha$ *be the largest number such that* $\det(A + vB) \neq 0$ *in the interval* $[0, v_\alpha)$. *Then, either* $v_\alpha = +\infty$, *or an element of* $Z(v)$ *must change sign in* $[0, v_\alpha)$.

*Proof.* As $v \longrightarrow v_\alpha$, at least one entry of $Z(v)$ must become infinite. Otherwise, in any interval $[0, v_\beta)$ where $Z(v) > 0$ we have $Z'(v) < 0$ (Lemma 1); therefore, $Z(v)$ is bounded in $[0, v_\beta)$,

$$0 < Z(v) \leq Z(0) = A^{-1}.$$

It follows that $v^* = \max v_\beta \leq v_\alpha$, with strict inequality if $v^* < +\infty$, because $0 \leq Z(v^*) \leq A^{-1}$. When $v^* < +\infty$, the thesis follows from $Z'(v^*) \leq 0$ and Lemma 1 (note that the entries $z_{ij}(v)$ cannot vanish identically). □

**Theorem 1.** *Let* $v^*$ *be the largest, possibly infinite, number such that* $Z(v) > 0$ *in* $[0, v^*)$. *Then* $v^*$ *is the limit of the sequence* $\{v_k\}$ *given by*

$$(6) \qquad v_{k+1} = v_k + \min_{i,j;\, w_{kij} > 0} z_{kij}/w_{kij}, \qquad k = 0, 1, 2, \ldots, n;\ v_0 = 0,$$

*where* $Z_k = Z(v_k) = [z_{kij}]$, $W_k = -Z'(v_k) = Z_k B Z_k = [w_{kij}]$.

*Proof.* Let $v^*_{ij}$ be the smallest value of $v$ for which $z_{ij}(v) = 0$, if such a value exists, or $+\infty$ otherwise. We have $v^* = \min_{i,j} v^*_{ij}$. In $[0, v^*)$, the matrix $Z(v)$ does not have singularities (Lemma 2) and its entries are strictly decreasing and convex functions of $v$ (Lemma 1). These regularity conditions on the entries $z_{ij}(v)$ allow us to obtain the sequence $\{v_k\}$, given by (6), as follows: we compute the Newton steps for the elements of the equation $Z(v) = 0$ and use the smallest of them to update $v$.

The first iteration, with starting value $v_0 = 0$, produces the equations $z_{ij}(0) + v z'_{ij}(0) = 0$, where $z_{ij}(0) > 0$ and $z'_{ij}(0) < 0$. The smallest solution of these equations is the first approximation $v_1$ in (6) and it is the largest value of $v$ for which

$$Z(0) + vZ'(0) = A^{-1} - vA^{-1}BA^{-1} \geq 0.$$

As $Z(v) > Z(0) + vZ'(0)$ for $0 < v < v^*$, we have $v_1 < v^*_{ij}$, $i, j = 1, 2, \ldots, n$; therefore, $0 < v_1 < v^*$ and $Z_1 > 0$, $W_1 > 0$.

The successive approximations $v_k$ are defined as follows. Suppose we have computed the approximation $v_k$, for some $k > 0$, for which we have $0 < v_k < v^*$, $Z_k > 0$, $W_k > 0$. We compute the Newton steps starting from the value $v_k$, common to all the equations $z_{ij}(v) = 0$; this produces the equations $z_{ij}(v_k) + (v - v_k)z'_{ij}(v_k) = 0$. The approximation $v_{k+1}$ (the smallest solution of these equations) is the largest value of $v$ for which

$$Z(v_k) + (v - v_k)Z'(v_k) = Z_k - (v - v_k)W_k \geq 0$$

and it is given by (6). As $Z(v) > Z(v_k) + (v - v_k)Z'(v_k)$ for $v_k < v < v^*$, we have $v_{k+1} < v^*_{ij}$, $i, j = 1, 2, \ldots, n$; therefore $v_k < v_{k+1} < v^*$ and $Z_{k+1} > 0$, $W_{k+1} > 0$. We conclude that the sequence $\{v_k\}$ is increasing, bounded from above by $v^*$ if $v^* < +\infty$, and convergent to $v^*$ (note that $\{v_k\}$ cannot converge to a limit $v^*_1 < v^*$ since this would imply $(v_{k+1} - v_k) \longrightarrow \min_{ij} z_{ij}(v^*_1)/|z'_{ij}(v^*_1)| > 0$).

When $v^* = +\infty$, all the entries $z_{ij}(v)$ are positive, strictly decreasing, and convex functions of $v \in [0, +\infty)$ (the only possible solution of each equation $z_{ij}(v) = 0$ is $v^* = +\infty$). If the sequence $\{v_k\}$ were bounded, then it would be convergent: $v_k \to v^*_1 < +\infty$; as above, we would have $(v_{k+1} - v_k) \to$ constant $> 0$. Thus, $\{v_k\}$ is not bounded and it is diverging monotonically to $+\infty$. □

*Remarks.* (a) It is possible to show that the sequence $\{v'_k\}$ given by

$$v'_{k+1} = v'_k + \min_{i, j;\, w_{0ij} > 0} z_{kij}/w_{0ij}, \qquad k = 0, 1, 2, \ldots; v'_0 = 0,$$

is convergent to $v^*$, if $v^* < +\infty$, or divergent to $+\infty$ otherwise.

(b) Only for very small $n$ (the first few integers) can we obtain the analytic expressions of the entries $z_{ij}(v)$ $(i, j = 1, 2, \ldots, n)$ and find their zeros to evaluate $v^*$. The application of the iterative process (6) involves the numerical computation of the inverses $Z_k$, and each iteration requires $O(n^3)$ operations; however, the method has been applied successfully with $n$ equal to 30, 40, and 50 (for example, by using matrices from one-dimensional boundary value problems).

(c) We can show the quadratic convergence [3, p. 260] of the process (6) when $v^* < +\infty$ and $Z'(v^*) > 0$. We introduce in (6) $z_{kij}$ obtained from Taylor's formula

$$z_{ij}(v^*) = z_{kij} - (v^* - v_k)w_{kij} + \tfrac{1}{2}(v^* - v_k)^2 z''_{ij}(v_{kij}),$$

where $v_k \le v_{kij} \le v^*$. After some manipulations we have

$$v^* - v_{k+1} = \min_{i,j;\,w_{kij}>0} \left[\tfrac{1}{2}(v^* - v_k)^2 z''_{ij}(v_{kij}) - z_{ij}(v^*)\right]/w_{kij};$$

thus, as $z_{ij}(v^*) \ge 0$ and $w_{kij} > 0$, it follows that

$$s_{k+1} \le \max_{i,j;\,w_{kij}>0} z''_{ij}(v_{kij})/w_{kij} \to \max_{i,j} z''_{ij}(v^*)/|z'_{ij}(v^*)|,$$

where

(7) $$s_{k+1} = (v^* - v_{k+1})/(v^* - v_k)^2.$$

## 3. The inverse of $C(v) = A + v(B - B^{\mathrm{T}})$

**Theorem 2.** *Let the symmetric matrix $A + A^{\mathrm{T}}$ be positive definite. Then, in $[0, v^*)$ the spectral radius $h$ of the nonnegative matrix*

(8) $$H(v) = vZ(v)B^{\mathrm{T}}$$

*is less than $1$, and $C^{-1}(v) \ge Z(v) > 0$.*

*Proof.* The matrix $C(v)$ given by (4) is now written as

$$C(v) = (A + vB)[I - H(v)],$$

where $H(v)$ is given by (8). In $[0, v^*)$ we have $Z(v) > 0$; it follows that $C^{-1}(v) \ge Z(v) > 0$ if the spectral radius $h(v) = r(H)$ of the nonnegative matrix $H(v)$ is less than 1 [5, p. 83]. To the spectral radius $h$ there corresponds an eigenvector $\mathbf{u} \ge 0$; from the eigenvalue equation $vB^{\mathrm{T}}\mathbf{u} = h(A + vB)\mathbf{u}$ we obtain

$$h = v\mathbf{u}^{\mathrm{T}}B\mathbf{u}/(\mathbf{u}^{\mathrm{T}}A\mathbf{u} + v\mathbf{u}^{\mathrm{T}}B\mathbf{u}).$$

We have $\mathbf{u}^{\mathrm{T}}A\mathbf{u} = \frac{1}{2}[\mathbf{u}^{\mathrm{T}}(A + A^{\mathrm{T}})\mathbf{u}] > 0$, because $A + A^{\mathrm{T}}$ is assumed positive definite. Thus, as $v \ge 0$, $\mathbf{u} \ge 0$, $B \ge 0$, it follows that $h < 1$. □

*Remarks.* By means of simple examples it is possible to show that:

(a) The condition $A + A^{\mathrm{T}}$ positive definite is not necessary to have $h(v) < 1$, $0 \le v < v^*$.

(b) The condition $H(v) \ge 0$, $0 \le v < v^*$, is not sufficient by itself to have $h(v) < 1$.

## 4. Numerical results

As a sample problem we use the matrix $A + vB$ obtained from a finite difference approximation to (5) using central differences and the trapezium rule.

Here we present the results obtained by assuming in (5) that $p = 1$, $q = 0$, and $K(x, x') = \exp(-(x - x')^2)$ (sample problem 1). In this case, $A$ is a Stieltjes matrix [5, p. 85] and $B$ is a positive matrix. The inverses $Z_k$ are computed by means of the routine LINV2F of the IMSL Library. The results (double-precision computation) are shown in Table 1. The quantities $s_k$, given by (7), tend to a constant value confirming quadratic convergence. Values of $v$ greater than $v^*$, for which some computed entries of $Z(v)$ are less than zero are reported in the row $*$.

TABLE 1
Values of $v_k$ and of $s_k$ for the sample problem 1 for different values of the mesh spacing $1/m$.

| | $m = 30$ | | $m = 40$ | | $m = 50$ | |
|---|---|---|---|---|---|---|
| $k$ | $v_k$ | $s_k$ | $v_k$ | $s_k$ | $v_k$ | $s_k$ |
| 0 | 0. | | 0. | | 0. | |
| 1 | 5.145497 | 0.0658 | 5.076475 | 0.0678 | 5.035965 | 0.0690 |
| 2 | 8.172000 | 0.0491 | 8.018864 | 0.0496 | 7.929456 | 0.0499 |
| 3 | 8.820264 | 0.0505 | 8.633288 | 0.0510 | 8.524586 | 0.0517 |
| 4 | 8.842959 | 0.0508 | 8.653839 | 0.0514 | 8.543958 | 0.0517 |
| 5 | 8.842985 | 0.0508 | 8.653861 | 0.0513 | 8.543978 | 0.0516 |
| 6 | 8.842985 | | 8.653861 | | 8.543978 | |
| * | 8.85 | | 8.66 | | 8.55 | |

TABLE 2
Values of $v_k$ and of $s_k$ for the sample problem 2 for different values of the mesh spacing $1/m$.

| | $m = 30$ | | $m = 40$ | | $m = 50$ | |
|---|---|---|---|---|---|---|
| $k$ | $v_k$ | $s_k$ | $v_k$ | $s_k$ | $v_k$ | $s_k$ |
| 0 | 0. | | 0. | | 0. | |
| 1 | 0.411695 | 1.8439 | 0.299168 | 2.5543 | 0.234883 | 3.2660 |
| 2 | 0.471457 | 0.2874 | 0.341517 | 0.3880 | 0.267634 | 0.4885 |
| 3 | 0.472521 | 0.2899 | 0.342236 | 0.3912 | 0.268175 | 0.4924 |
| 4 | 0.472521 | 0.2881 | 0.342236 | 0.3884 | 0.268175 | 0.5178 |
| 5 | 0.472521 | | 0.342236 | | 0.268175 | |
| * | 0.48 | | 0.35 | | 0.27 | |
| ** | 0.49 | | 0.36 | | 0.28 | |

Now we consider the matrix $C(v) = A + v(B - B^{\mathrm{T}})$ obtained by assuming in (5) that $p = 1$, $q = 0$, and $K(x, x') = x - x'$ (sample problem 2). Here the matrix $B$ is the nonnegative contribution due to $K(x, x')$ for $x \geq x'$. The results are shown in Table 2. Values of $v$ greater than $v^*$, for which some computed entries of $Z(v)$ and of $C^{-1}(v)$ are less than zero are reported in the rows $*$ and $**$, respectively. We note that $[0, v^*)$ is a sufficiently good approximation of the interval in which $C^{-1}(v) > 0$.

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## Bibliography

1. G. Buffoni and A. Galati, *Matrici essenzialmente positive con inversa positiva*, Boll. Un. Mat. Ital. (4) **10** (1974), 98–103.
2. K. Fan, *Topological proofs for certain theorems on matrices with non-negative elements*, Monatsh. Math. **62** (1958), 219–237.
3. J. R. Rice, *Numerical methods, software and analysis*, McGraw-Hill, 1983.
4. Y. M. Svirezhev and D. O. Logofet, *Stability of biological communities*, MIR, Moscow, 1983.
5. R. S. Varga, *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, N. J. 1965.

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